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A Noncollision Periodic Solution for N-Body Problems[1]

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Using variational minimization methods, we prove the existence of one noncollision periodic solution for N-body type problems whose potentials are pinched between two homogeneous potentials in $\mathbf{R}^k (k \geq 2)$. © 2001 Academic Press

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1. INTRODUCTION AND MAIN RESULTS

The motion of N-body type problems [1, 2, 9, 12, 19, 20] is related with solving the following second order differential equations,

$$m_i \ddot{q}_i = \frac{\partial U}{\partial q_i}, \tag{1.1}$$

where $m_i > 0$ is the mass of the ith body and $q_i \in \mathbf{R}^k (k \geq 2)$ is the position of the ith body, and the potential

$$U(q) = U(q_1, \ldots, q_N) = \sum_{1 \leq i < j \leq N} U_{ij}(q_i - q_j), \tag{1.2}$$

where $U_{ij}(x) \in C^1(\mathbf{R}^k \backslash 0, \mathbf{R})$.

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In the last 20 years, some researchers applied variational methods to study the periodic solutions of N-body type problems [3–8, 10, 13, 14, 18, 21, 22], but they didn't get the existence of one noncollision periodic solution for any given masses of N bodies. Observing the symmetry and choosing a suitable domain of the Lagrangian action integral for (1.1), we prove that the minimizer of the Lagrangian action integral is one noncollision periodic solution of (1.1)–(1.2) assuming the potential $U(q)$ is pinched between two homogeneous potentials.

Let $O(k)(k \geq 2)$ denote the rotational group in $\mathbf{R}^k$ and

$$A(\theta) = \begin{pmatrix} B(\theta) & 0 \\ 0 & -I_{k-2} \end{pmatrix} \in O(k) \qquad \theta \in (0, 2\pi), \tag{1.3}$$

where I_{k-2} is a unit matrix with order $k-2$ and

$$B(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{1.4}$$

Let

$$H = W^{1,2}\left(\mathbf{R}/_{TZ}, \mathbf{R}^k\right) \tag{1.5}$$

$$H_{\#} = \left\{ x \in H \middle| x\left(t + \frac{T}{r}\right) = A\left(\frac{2\pi}{r}\right)x(t), r \geq 2 \text{ an integer} \right\} \tag{1.6}$$

$$E = \left\{q = (q_1, \ldots, q_N) \middle| q_i - q_j \in H_{\#}, i, j = 1, \ldots, N\right\}$$

$$\widetilde{E} = \left\{ q = (q_1, \ldots, q_N) \in E \middle| \sum_{i=1}^{N} m_i q_i(t) \equiv 0 \right\} \tag{1.7}$$

$$\Lambda = \left\{ q = (q_1, \ldots, q_N) \in \widetilde{E} \middle| q_i(t) \neq q_j(t), \forall\, t \in \mathbf{R}, 1 \leq i \neq j \leq N \right\} \tag{1.8}$$

$$f(q) = \frac{1}{2}\int_0^T \sum_{i=1}^{N} m_i |\dot{q}_i|^2 dt + \int_0^T U(q)dt. \tag{1.9}$$

THEOREM 1.1. *Assume $U(q)$ satisfies*

(1)

$$\frac{a}{2}\sum_{1\leq i<j\leq N} \frac{m_i m_j}{|q_i - q_j|^{\alpha}} \leq U(q) \leq \frac{b}{2}\sum_{1\leq i<j\leq N} \frac{m_i m_j}{|q_i - q_j|^{\alpha}}, \qquad \alpha > 0 \tag{1.10}$$

(2)

$$U(Aq) = U(q), \tag{1.11}$$

for some

$$A = A(2\pi/r) \in O(k),$$

where r will be defined later.

Then there is an integer r depending on α *and masses* $m_1, \ldots, m_N$ *such that the minimizer of* $f(q)$ *on* $\overline{\Lambda}$ *is one noncollision* T*-periodic solution for* (1.1)–(1.2).

THEOREM 1.2. *If* $\alpha = 1$, $N = 3$, $m_1 = m_2 = m_3 = 1$, *and* $a = b = 1$ *then the minimizer of* $f(q)$ *on* $\overline{\Lambda}$ *with* $r = 2$ *is one noncollision* T*-periodic solution.*

Remark. The domain Λ for $f(q)$ is different from the one defined by Bessi and Coti Zelati [4].

2. THE PROOF OF THEOREM 1.1

In order to prove Theorem 1.1, first we give some lemmas:

LEMMA 2.1. *The critical points of* $f(q)$ *in* Λ *are noncollision* T*-periodic solutions of* (1.1)–(1.2).

First, we prove that the critical point $q \in \widetilde{E}$ for $f(q)$ restricted on $\widetilde{E}$ is also a critical point for $f(q)$ on E.

In fact, the condition that the center of masses is fixed at the origin is equivalent to

$$g(q) = \left| \sum_{i=1}^{N} m_i q_i(t) \right|^2 \equiv 0. \tag{2.1}$$

Hence by the Lagrangian multiplier rule, for any critical point q of $f(q)$ on $\widetilde{E}$, we have

$$f'(q) + \lambda g'(q) = 0; \tag{2.2}$$

that is, for any $\varphi = (\varphi_1, \ldots, \varphi_N)$, $\varphi_i \in W^{1,2}(\mathbf{R}/_{TZ}, \mathbf{R}^k)$ we have

$$\langle f'(q), \varphi\rangle + \lambda \langle g'(q), \varphi\rangle = 0 \tag{2.3}$$

$$\langle f'(q), \varphi\rangle + 2\lambda \left| \sum_{i=1}^{N} m_i q_i \right| \sum_{i=1}^{N} \langle m_i \dot{q}_i, \varphi_i\rangle = 0, \tag{2.4}$$

that is,

$$\langle f'(q), \varphi \rangle = 0. \tag{2.5}$$

Now assume $q \in E$ is a critical point of $f(q)$ on E. Then

$$\langle f'(q), y \rangle = 0, \qquad \forall\, y \in E. \tag{2.6}$$

Hence we have $p = (p_1, \dots, p_N) \in E^{\perp}$, where

$$p_i \equiv \ddot{q}_i - \frac{\partial U}{\partial q_i}, \qquad i = 1, \dots, N. \tag{2.7}$$

On the other hand, we notice that

$$\begin{aligned} p_i - p_j &= \ddot{q}_i - \ddot{q}_j - \frac{\partial U}{\partial q_i} + \frac{\partial U}{\partial q_j} \\ &= (q_i - q_j)'' - \sum_{j \neq i, j=1}^{N} \frac{a\alpha m_i m_j (q_j - q_i)}{|q_i - q_j|^{\alpha+2}} \\ &\quad + \sum_{i \neq j, i=1}^{N} \frac{a\alpha m_i m_j (q_i - q_j)}{|q_j - q_i|^{\alpha+2}}. \end{aligned} \tag{2.8}$$

Hence $p_i - p_j \in H_{\#}$, $p \in E$. Hence $p \in E^{\perp} \cap E = \{0\}$; that is, q is a solution of (1.1).

LEMMA 2.2. *The functional f is coercive on Λ; that is, for any $\{q_n\} \sqsubset \Lambda$, $\|q_n\|_H \to +\infty$, $f(q_n) \to +\infty$.*

Proof. For any $q = (q_1, \dots, q_N) \in \Lambda$, we have

$$\begin{aligned} (q_i - q_j)\left(t + \frac{T}{r}\right) &\equiv q_i\left(t + \frac{T}{r}\right) - q_j\left(t + \frac{T}{r}\right) \\ &= A\left(\frac{2\pi}{r}\right)(q_i(t) - q_j(t)) \\ &\equiv A\left(\frac{2\pi}{r}\right)(q_i - q_j)(t) \end{aligned} \tag{2.9}$$

$$\begin{aligned} &\left|(q_i - q_j)\left(t + \frac{T}{t}\right) - (q_i - q_j)(t)\right|^2 \\ &\quad = \left|A\left(\frac{2\pi}{r}\right)(q_i - q_j)(t) - (q_i - q_j)(t)\right|^2 \\ &\quad = \left|2 \sin \frac{\pi}{r}\right|^2 \left|(q_i - q_j)(t)\right|^2. \end{aligned} \tag{2.10}$$

On the other hand,

$$\begin{aligned}\left|(q_i - q_j)\left(t + \frac{T}{r}\right) - (q_i - q_j)(t)\right|^2 &= \left|\int_t^{t+\frac{T}{r}} (\dot{q}_i - \dot{q}_j)dt\right|^2 \\ &\leq \frac{T}{r}\left(\int_t^{t+\frac{T}{r}} |(\dot{q}_i - \dot{q}_j)(t)|^2 dt\right) \\ &= \frac{T}{r^2}\int_0^T \left|(\dot{q}_i - \dot{q}_j)(t)\right|^2 dt. \end{aligned} \tag{2.11}$$

Hence we have

$$\int_0^T \left|(\dot{q}_i - \dot{q}_j)(t)\right|^2 dt \geq \frac{r^2}{T}\left|2\sin\frac{\pi}{r}\right|^2 \left|(q_i - q_j)(t)\right|^2 \tag{2.12}$$

$$\int_0^T \sum_{1\leq i<j\leq N} m_i m_j \left|\dot{q}_i - \dot{q}_j\right|^2 dt \geq \frac{r^2}{T}\left|2\sin\frac{\pi}{r}\right|^2 \sum_{1\leq i<j\leq N} m_i m_j |q_i - q_j|^2 \tag{2.13}$$

$$M\int_0^T \sum_{i=1}^N m_i \left|\dot{q}_i\right|^2 dt \geq \frac{r^2}{T}\left|2\sin\frac{\pi}{r}\right|^2 M\sum_{i=1}^N m_i |q_i|^2, \tag{2.14}$$

where

$$M = \sum_{i=1}^N m_i. \tag{2.15}$$

Hence the standard norm for H is equivalent to

$$\|\dot{q}\|_2 = \left(\int_0^1 \sum_{i=1}^N m_i |\dot{q}_i|^2 dt\right)^{1/2}. \tag{2.16}$$

Hence the definition of $f(q)$ implies f is coercive.

LEMMA 2.3. *The system* (1.1)–(1.2) *has a weak* T-*periodic solution* $q = (q_1, \ldots, q_N) \in \overline{\Lambda}$ *in the sense of Bari and Rabinowitz* [3]:

(1°) $q_i \in W^{1,2}(\mathbf{R}/_{TZ}, \mathbf{R}^k)$.

(2°) *The collision set* $C = \{t \in [0, T] | q_i(t) = q_j(t)$ *for some* $1 \leq i \neq j \leq N\}$ *has Lebesgue measure* 0.

(3°) *q_i is C^2 on $[0, T]\backslash C$ and satisfies* (1.1) *and energy conservation*,

$$\frac{1}{2}\sum_{i=1}^{N} m_i|\dot{q}_i|^2 - U(q_1, \ldots, q_N) = h. \tag{2.17}$$

Proof. It's easy to prove $f(q)$ has positive lower bound and is weakly lower semi-continuous, so Lemma 2.2 implies Lemma 2.3.

In order to get a good lower bound estimate of $f(q)$ on the collision solutions, we need another lemma:

LEMMA 2.4. *Let index sets A and B satisfy $A \bigcap B = \emptyset$ and $A \bigcup B = \{1, 2, \ldots, N\}$. Then*

$$\sum_{(i,j)\in A\times B} \frac{m_i m_j}{|q_i - q_j|^\alpha}$$
$$\geq \left(\sum_{(i,j)\in(A\times B)} m_i m_j\right)^{1+\frac{\alpha}{2}} \left(\sum_{(i,j)\in(A\times B)} m_i m_j |q_i - q_j|^2\right)^{-\frac{\alpha}{2}}. \tag{2.18}$$

For the proof of Lemma 2.4 refer to Long and Zhang [13]. In order to facilitate the reader, we repeat it. By Hölder's inequality, we have

$$\left(\sum_{i\in A,\, j\in B} m_i m_j\right)^2 \leq \left(\sum_{i\in A,\, j\in B} \frac{m_i m_j}{|q_i - q_j|^\alpha}\right)\left(\sum_{i\in A,\, j\in B} m_i m_j |q_i - q_j|^\alpha\right). \tag{2.19}$$

By Hölder's inequality, we have

$$\sum_{i\in A,\, j\in B} m_i m_j |q_i - q_j|^\alpha$$
$$\leq \left(\sum_{i\in A,\, j\in B} m_i m_j\right)^{\frac{2-\alpha}{2}} \left(\sum_{i\in A,\, j\in B} m_i m_j |q_i - q_j|^2\right)^{\alpha/2}. \tag{2.20}$$

By (2.19) and (2.20) we get (2.18).

Now we estimate the lower bound of $f(q)$ on the collision solutions. Let S_N denote the group of all the permutations of $\{1, \ldots, N\}$. For $l = 2, \ldots, N$, we set

$$\partial\Lambda_l = \left\{q \in E \,|\, \exists\, s \in S_N, \exists\, \bar{t} \in [0, T] \text{ s.t. } q_{s(1)}(\bar{t}) = \cdots = q_{s(l)}(\bar{t})\right\}. \tag{2.21}$$

First, we assume $l = 2$, s is the identity, and $\bar{t} = 0$. Then by the Lagrangian identity and the symmetry property $(q_i - q_j)(t + \frac{T}{r}) = A(\frac{2\pi}{r})(q_i - q_j)(t)$ we have

$$f(q) \geq g_1(q) + g_2(q) + g_3(q), \tag{2.22}$$

where

$$g_1(q) = r\Bigg[\frac{1}{2M}\sum_{1\le i\ne j\le 2} m_i m_j \times \int_0^{T/r}\left(\frac{1}{2}|\dot q_i-\dot q_j|^2 + Ma\frac{1}{|q_i-q_j|^\alpha}\right)dt\Bigg] \tag{2.23}$$

$$g_2(q) = r\Bigg[\frac{1}{2M}\sum_{3\le i\ne j\le N} m_i m_j \times \int_0^{T/r}\left(\frac{1}{2}|\dot q_i-\dot q_j|^2 + Ma\frac{1}{|q_i-q_j|^\alpha}\right)dt\Bigg] \tag{2.24}$$

$$g_3(q) = r\Bigg[\frac{2}{2M}\sum_{1\le i\le 2,\, 3\le j\le N} m_i m_j \times \int_0^{T/r}\left(\frac{1}{2}|\dot q_i-\dot q_j|^2 + Ma\frac{1}{|q_i-q_j|^\alpha}\right)dt\Bigg]. \tag{2.25}$$

Using the estimates of the Lagrangian action integral on collision solutions of two body [7] problems we have

LEMMA 2.5.

$$g_1(q) \ge C_1 r^{\frac{2\alpha}{2+\alpha}} T^{\frac{2-\alpha}{2+\alpha}}, \tag{2.26}$$

where

$$C_1 = AM_2^{\frac{-\alpha}{2+\alpha}} \sum_{1\le i\ne j\le 2} m_i m_j \tag{2.27}$$

$$M_2 = \sum_{i=1}^{2} m_i \tag{2.28}$$

$$A = \frac{1}{2}\left(\frac{1}{2}+\frac{1}{\alpha}\right)(\alpha a)^{2/(\alpha+2)}(2\pi)^{\frac{2\alpha}{2+\alpha}}. \tag{2.29}$$

Let

$$B_2 = \min_{s\in S_N}\frac{\sum\limits_{1\le i\ne j\le 2} m_{s(i)}m_{s(j)}}{\left(\sum\limits_{i=1}^{2} m_{s(i)}\right)^{\alpha/(2+\alpha)}} \tag{2.30}$$

$$\widetilde{C_1} = AB_2. \tag{2.31}$$

Then

$$\inf\{g_1(q), q\in\partial\Lambda_2\} \ge \widetilde{C_1} r^{\frac{2\alpha}{2+\alpha}} T^{\frac{2-\alpha}{2+\alpha}}. \tag{2.32}$$

By the arguments of Degiovanni and Giannoni [7], we can get the lower bound estimate on $g_2(q)$,

$$g_2(q) \geq C_2 T^{\frac{2-\alpha}{2+\alpha}}, \tag{2.33}$$

where

$$C_2 = A M_{N-2}^{\frac{-\alpha}{2+\alpha}} \sum_{i=3}^{N} m_i m_j \tag{2.34}$$

$$M_{N-2} = \sum_{i=3}^{N} m_i. \tag{2.35}$$

Let

$$B_{N-2} = \min_{s \in S_N} \frac{\sum_{3 \leq i \neq j \leq N} m_{s(i)} m_{s(j)}}{(\sum_{i=3}^{N} m_{s(i)})^{\alpha/(2+\alpha)}} \tag{2.36}$$

$$\widetilde{C}_2 = A B_{N-2}. \tag{2.37}$$

Then

$$\inf\{g_2(q), q \in \partial\Lambda_2\} \geq \widetilde{C}_2 T^{\frac{2-\alpha}{2+\alpha}}. \tag{2.38}$$

We use inequality (2.18) of Lemma 2.4, Sundman's inequality [19], and the arguments of Degiovanni and Giannoni [7] to estimate the lower bound for $g_3(q)$:

$$\begin{aligned} g_3(q) \geq & \frac{1}{2M} \int_0^T \sum_{1 \leq i \leq 2,\, 3 \leq j \leq N} m_i m_j |\dot{q}_i - \dot{q}_j|^2 dt + a \Bigg(\sum_{1 \leq i \leq 2,\, 3 \leq j \leq N} m_i m_j \Bigg)^{1+\frac{\alpha}{2}} \\ & \times \Bigg(\sum_{1 \leq i \leq 2,\, 3 \leq j \leq N} m_i m_j |q_i - q_j|^2 \Bigg)^{-\alpha/2} dt \end{aligned} \tag{2.39}$$

$$\begin{aligned} \geq & \frac{1}{2M} \int_0^T \left| \frac{d}{dt} \Bigg[\Bigg(\sum_{1 \leq i \leq 2,\, 3 \leq j \leq N} m_i m_j |q_i - q_j|^2 \Bigg)^{1/2} \Bigg] \right|^2 dt \\ & + a \Bigg(\sum_{1 \leq i \leq 2,\, 3 \leq j \leq N} m_i m_j \Bigg)^{1+\frac{\alpha}{2}} \\ & \times \int_0^T \Bigg(\sum_{1 \leq i \leq 2,\, 3 \leq j \leq N} m_i m_j |q_i - q_j|^2 \Bigg)^{-\alpha/2} dt \end{aligned} \tag{2.40}$$

$$\begin{aligned} \geq \inf \Bigg\{ & \frac{1}{2M} \int_0^T |\dot{r}|^2 dt + a \Bigg(\sum_{1 \leq i \leq 2,\, 3 \leq j \leq N} m_i m_j \Bigg)^{1+\frac{\alpha}{2}} \\ & \times \int_0^T r^{-\alpha} dt \,|\, r \in W^{1,2}([0,T], R^+) \Bigg\} \end{aligned} \tag{2.41}$$

$$= T\inf\left\{\frac{1}{2M}\left(\frac{2\pi}{T}\right)^2 R^2 + a\left(\sum_{1\le i\le 2,\,3\le j\le N} m_i m_j\right)^{1+\frac{\alpha}{2}} \times \frac{1}{R^\alpha}, R>0\right\} \tag{2.42}$$

$$= \left(\frac{\alpha}{2}\right)^{\frac{2}{\alpha+2}}\left(1+\frac{2}{\alpha}\right)\left[\frac{1}{2M}\left(\frac{2\pi}{T}\right)^2\right]^{\frac{\alpha}{\alpha+2}}\left[\left(a\sum_{1\le i\le 2,\,3\le j\le N} m_i m_j\right)^{1+\frac{\alpha}{2}}\right]^{\frac{2}{\alpha+2}}$$

$$= \left(\frac{\alpha}{2}\right)^{\frac{2}{\alpha+2}}\left(1+\frac{2}{\alpha}\right)(2\pi^2)^{\frac{\alpha}{\alpha+2}} M^{-\frac{\alpha}{\alpha+2}} a^{\frac{2}{\alpha+2}}\left(\sum_{1\le i\le 2,\,3\le j\le N} m_i m_j\right) T^{\frac{2-\alpha}{2+\alpha}}$$

$$= \frac{1}{2}\left(1+\frac{2}{\alpha}\right)(\alpha a)^{\frac{2}{\alpha+2}}(2\pi)^{\frac{2\alpha}{\alpha+2}} M^{-\frac{\alpha}{\alpha+2}}\left(\sum_{1\le i\le 2,\,3\le j\le N} m_i m_j\right) T^{\frac{2-\alpha}{2+\alpha}}$$

$$= 2AM^{-\frac{\alpha}{\alpha+2}}\left(\sum_{1\le i\le 2,\,3\le j\le N} m_i m_j\right) T^{\frac{2-\alpha}{2+\alpha}} \equiv C_3 T^{\frac{2-\alpha}{2+\alpha}}. \tag{2.43}$$

Let

$$B = \frac{\min_{s\in S_N}\sum_{1\le i\le 2,\,3\le j\le N} m_{s(i)} m_{s(j)}}{M^{\alpha/(\alpha+2)}} \tag{2.44}$$

$$\widetilde{C}_3 = 2AB. \tag{2.45}$$

Then

$$\inf\{g_3(q), q\in\partial\Lambda_2\} \ge \widetilde{C}_3 T^{\frac{2-\alpha}{2+\alpha}} \tag{2.46}$$

$$\inf\{f(q), q\in\partial\Lambda_2\} \ge (\widetilde{C}_1 r^{\frac{2\alpha}{2+\alpha}} + \widetilde{C}_2 + \widetilde{C}_3) T^{\frac{2-\alpha}{2+\alpha}}. \tag{2.47}$$

It's easy to see that for $l > 2$ we also have

$$\inf\{f(q), q\in\partial\Lambda_l\} \ge (\widetilde{C}_1 r^{\frac{2\alpha}{2+\alpha}} + \widetilde{C}_2 + \widetilde{C}_3) T^{\frac{2-\alpha}{2+\alpha}}. \tag{2.48}$$

Remark. The corresponding lower bound estimate in Bessi and Coti Zelati [4] is not correct since the symmetry breaks down when they move the binary collision to the origin and they work on their domain.

LEMMA 2.6 [6]. *Let*

$$\rho = \sum_{1\le i\ne j\le N} \frac{m_i m_j}{|\sin(\pi(i-j))/N|^\alpha} \tag{2.49}$$

$$\sigma = \sum_{1\le i\ne j\le N} m_i m_j\left|\sin\frac{\pi(i-j)}{N}\right|^2. \tag{2.50}$$

Then the minimizing value for Lagrangian action $f(q)$ has upper bound estimate

$$
\begin{aligned}
f(q) &\leq \frac{1}{2}\left(\frac{1}{2}+\frac{1}{\alpha}\right)(b\alpha)^{\frac{2}{2+\alpha}}(2\pi)^{\frac{2\alpha}{\alpha+2}}\rho^{\frac{2}{\alpha+2}}\sigma^{\frac{\alpha}{\alpha+2}}M^{-\alpha/(\alpha+2)}T^{\frac{2-\alpha}{2+\alpha}} \\
&\equiv \widetilde{C}T^{\frac{2-\alpha}{2+\alpha}}.
\end{aligned}
\tag{2.51}
$$

Now we can prove Theorem 1.1.

Assume the minimizer q for $f(q)$ on $\overline{\Lambda}$ has a collision time $\bar{t} \in [0, T]$. Then by Lemma 2.5, we can get a lower bound estimate $(\widetilde{C_1}r^{2\alpha/(2+\alpha)} + \widetilde{C_2} + \widetilde{C_3})T^{(2-\alpha)/(2+\alpha)}$ for $f(q)$, which depends on the integer $r \geq 2$. By Lemma 2.6 we can choose r large enough so that

$$
\widetilde{C_1}r^{\frac{2\alpha}{2+\alpha}} + \widetilde{C_2} + \widetilde{C_3} > \widetilde{C}. \tag{2.52}
$$

This is a contradication.

Now we prove Theorem 1.2.

If $\alpha = 1, m_1 = m_2 = m_3 = 1, a = 1$, and $r = 2$ then we have

$$
\begin{aligned}
A &= \frac{3}{4}(2\pi)^{2/3} \\
B_2 &= 2^{2/3} \\
\widetilde{C_1} &= 3 \cdot 2^{-4/3}(2\pi)^{2/3}, \qquad \widetilde{C_1}r^{\frac{2\alpha}{2+\alpha}} = 3 \cdot 2^{-2/3}(2\pi)^{2/3} \\
\widetilde{C_2} &= 0 \\
B &= 2 \cdot 3^{-1/3} \\
\widetilde{C_3} &= 2AB = 3^{2/3}(2\pi)^{2/3} \\
\widetilde{C_1}r^{\frac{2\alpha}{2+\alpha}} + \widetilde{C_2} + \widetilde{C_3} &= \left(3 \cdot 2^{-2/3} + 3^{2/3}\right)(2\pi)^{2/3}.
\end{aligned}
$$

On the other hand, we compute

$$
\begin{aligned}
\rho &= \sum_{1 \leq i \neq j \leq 3} \frac{1}{|\sin(\pi(i-j))/3|} = \frac{2}{\sqrt{3}}6 = 4 \cdot 3^{1/2} \\
\sigma &= \sum_{1 \leq i \neq j \leq 3} \left|\sin\frac{\pi(i-j)}{3}\right|^2 = 6\left|\sin\frac{\pi}{3}\right|^2 = \frac{9}{2} \\
\widetilde{C} &= \frac{1}{2}\frac{3}{2}(2\pi)^{2/3}(4 \cdot 3^{1/2})^{2/3}\left(\frac{9}{2}\right)^{1/3} \cdot 3^{-1/3} \\
&= \frac{1}{2}3^{5/3}(2\pi)^{2/3} < (3 \cdot 2^{-2/3} + 3^{2/3})(2\pi)^{2/3} \\
&= \widetilde{C_1}r^{\frac{2\alpha}{2+\alpha}} + \widetilde{C_2} + \widetilde{C_3}.
\end{aligned}
$$

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